Karger's Contraction Algorithm

Introduction to Big Data Algorithms

Sebastian Forster

University of Salzburg

Global Minimum Cut

Definition

A cut of a graph $G = (V, E)$ with edge capacities is a non-trivial partition (S, T) of the nodes (i.e., $S \cup T = V$, $S \cap T = \emptyset$, and $\emptyset \subset S \subset V$). The capacity of a cut (S, T) is $c(S, T) := \sum_{(u,v) \in E : u \in S, v \in T} c(u, v)$.

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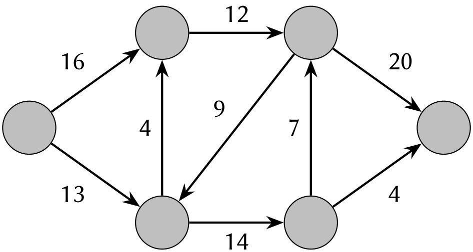

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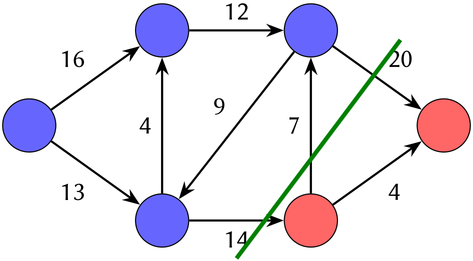
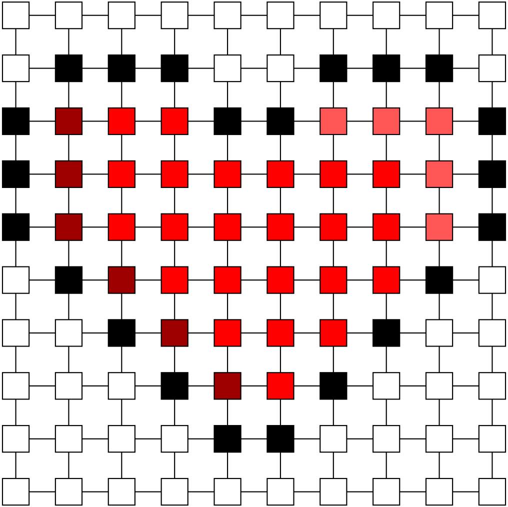


Example: Image Segmentation

Image as **grid graph**



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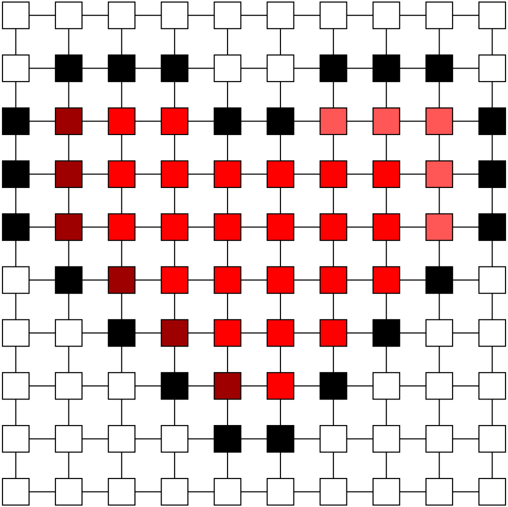

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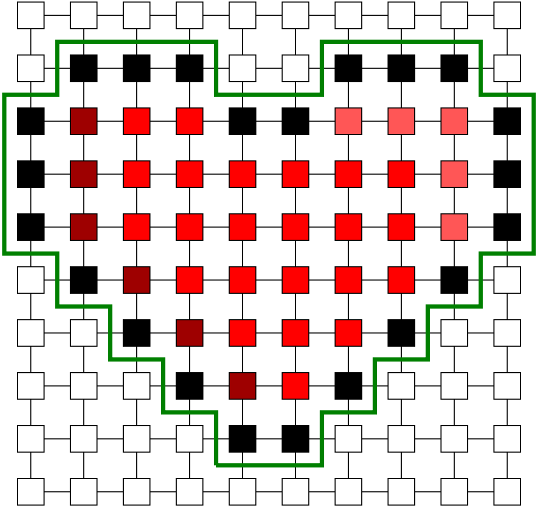

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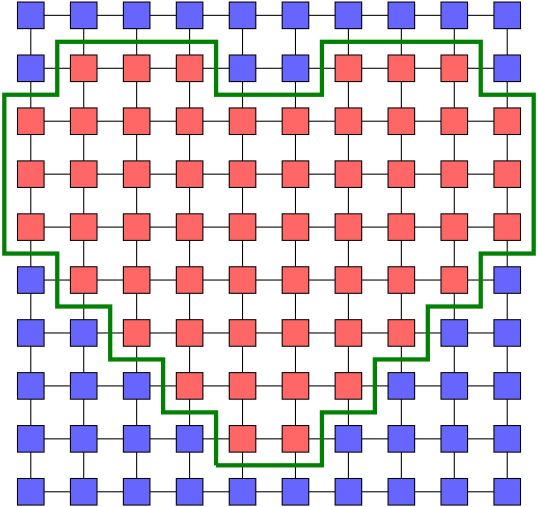

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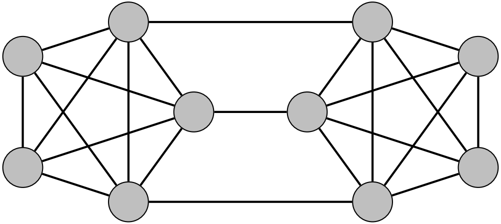

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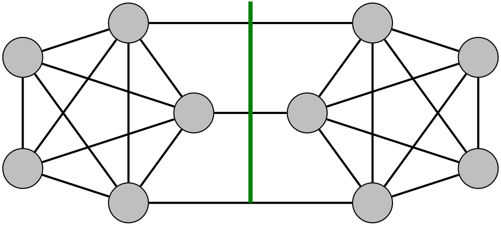

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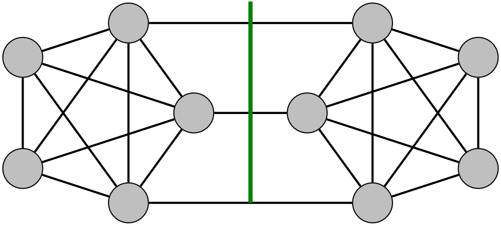

Observation

In an unweighted graph, the edge connectivity equals the capacity of the minimum cut ($\lambda = \Gamma^*$).

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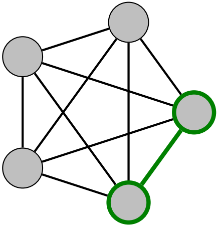

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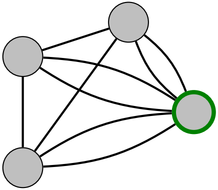

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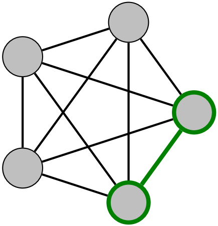

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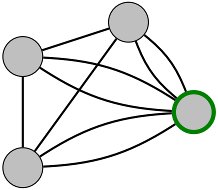

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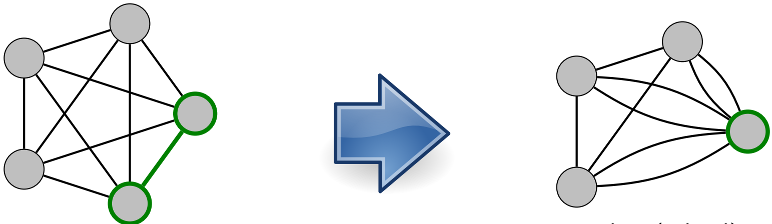

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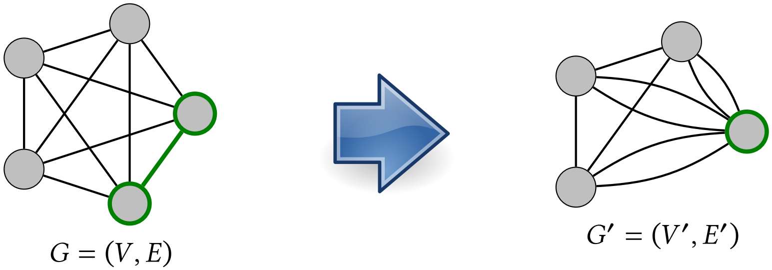

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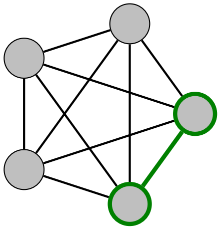

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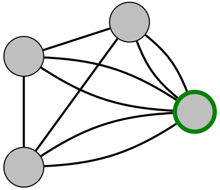

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Running time: $O(n)$ per contraction

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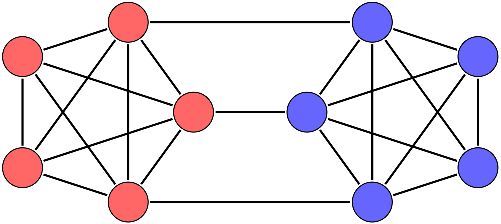
- Running time $O(n)$ per iteration
- Thus: total running time $O(n^2)$

Example



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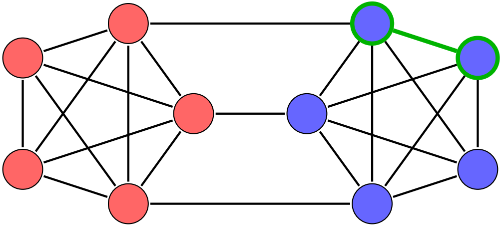

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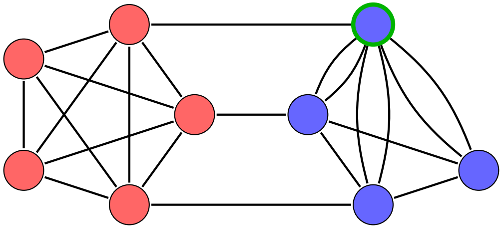

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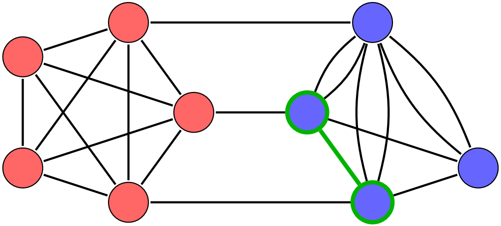

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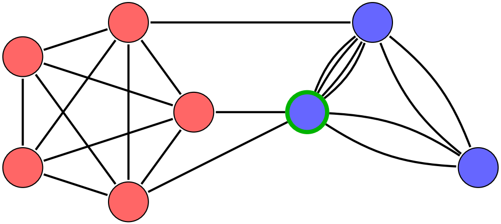

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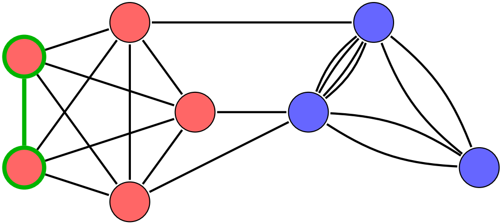

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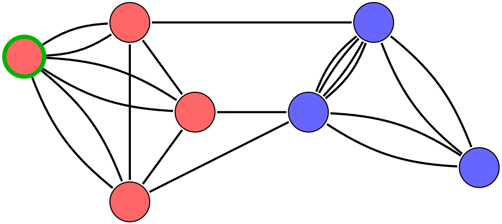

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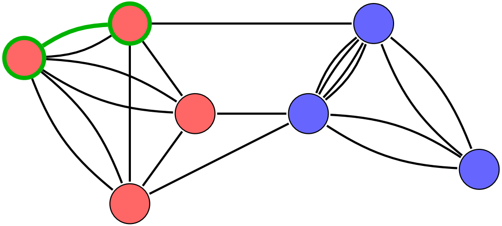

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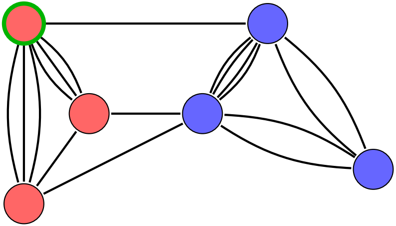

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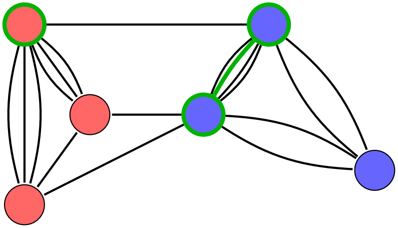

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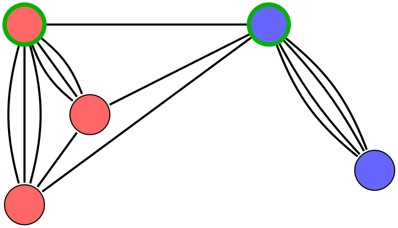

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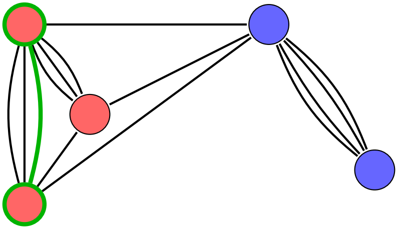

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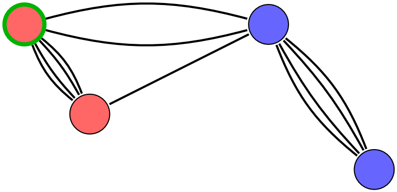

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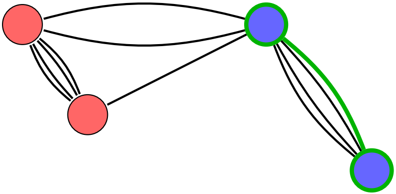

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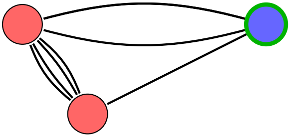

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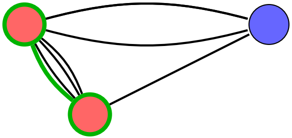

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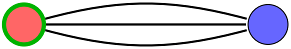

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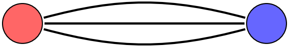

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(Probability $\leq \frac{2}{n'}$ for graph with n' nodes)

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&= \frac{2}{n \cdot (n-1)}
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Important Inequality

$(1-x)^{\frac{1}{x}} \leq \frac{1}{e}$ for $x \geq 1$

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Important Inequality

$(1 - x)^{\frac{1}{x}} \leq \frac{1}{e}$ for $x \geq 1$

In general: Error probability with $r = \left\lceil \frac{n(n-1)}{2} \ln \frac{1}{p} \right\rceil$ repetitions:

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Typical: Error probability of $\frac{1}{n^2}$ requires $O(n^2 \log n)$ repetitions

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Theorem

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- More efficient approach using this idea: running time $O(n^2 \log^3 n)$ [Karger/Stein '96]

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Remarks:

- More efficient approach using this idea: running time $O(n^2 \log^3 n)$ [Karger/Stein '96]
- Contraction idea does not generalize to directed graphs

Questions