Discrete Probability Theory

Introduction to Big Data Algorithms

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Discrete Probability Spaces

Discrete Probability Space (Ω, p)

- Countable **sample space** Ω consisting of "outcomes" ("simple events")
- **Probability mass function** $p : \Omega \to [0, 1]$ with $\sum_{\omega \in \Omega} p(\omega) = 1$

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$\Pr[O] = 1 - \Pr[\bar{O}] = 1 - \Pr[E] = 1 - \frac{1}{2} = \frac{1}{2}$

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If $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists, then we say that the series **converges** and $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ called the **value** of the series.

Boole's Inequality

Lemma (Boole's Inequality aka Union Bound)

Let A und B be two events and let $C = A \cup B$ be the event that occurs if A occurs B occurs. Then it holds that $\Pr[C] \leq \Pr[A] + \Pr[B]$.
In general, for every finite, nonempty set of events $\mathcal{A} = \{A_1, \ldots, A_n\}$ it holds that: $\Pr[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} \Pr[A_i]$.

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Actually: $O \cup S = \{1, 2, 4, 6\}$ and thus $\Pr[O \cup S] = \frac{4}{6} = \frac{2}{3}$

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Thus: E and S are independent.

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Theorem (Law of Total Probability)

For any events A and B it holds that

$$\Pr[A] = \Pr[A \cap B] + \Pr[A \cap \bar{B}] = \Pr[A \mid B] \cdot \Pr[B] + \Pr[A \mid \bar{B}] \cdot \Pr[\bar{B}]$$

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$$\Pr[P] = \Pr[P \mid E] \cdot \Pr[E] + \Pr[P \mid O] \cdot \Pr[O] = \left(\frac{1}{3} + \frac{2}{3}\right) \cdot \frac{1}{2} = \frac{1}{2}$$

Bayes' Theorem

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Sometimes: Obtain $\Pr[B]$ by law of total probability

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$$\mathrm{Ex}[X] = 1 \cdot \Pr[X = 1] + \cdots + 6 \cdot \Pr[X = 6] = (1 + \cdots + 6)\frac{1}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$$

Linearity of Expectation

Theorem (Linearity of Expectation)

Let X and Y be random variables and let $Z := X + Y$ be the sum of these random variables. Then it holds that $\mathrm{Ex}[Z] = \mathrm{Ex}[X] + \mathrm{Ex}[Y]$.
In general: For any finite sequence of random variables $X_1, \ldots, X_n$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ it holds that $\mathrm{Ex}[\sum_{i=1}^{n} \alpha_i X_i] = \sum_{i=1}^{n} \alpha_i \mathrm{Ex}[X_i]$.

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$\mathrm{Ex}[X + Y] = \mathrm{Ex}[X] + \mathrm{Ex}[Y] = 3.5 + 3.5 = 7.$

Variance

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Example

Die roll: $\Omega = \{1, 2, 3, 4, 5, 6\}$, $p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \frac{1}{6}$

$$X(\omega) := \omega$$

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$$\mathrm{Ex}[X] = \frac{7}{2} \qquad \mathrm{Var}[X] = \sum_{i=1}^{6} \frac{1}{6} \left(i - \frac{7}{2}\right)^2 = \frac{35}{12} \approx 2.92$$

Uniform Distribution

A random variable X has a **uniform distribution** with parameters $a \in \mathbb{N}$ and $b \in \mathbb{N}$, where $a \leq b$, ($X \sim U(a, b)$) if

$$\Pr[X = k] = \begin{cases} \frac{1}{n} & \text{if } k \in \{a, a+1, \ldots, b\} \\ 0 & \text{otherwise} \end{cases}$$

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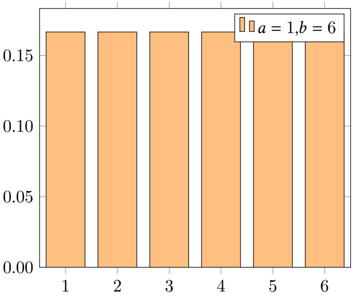

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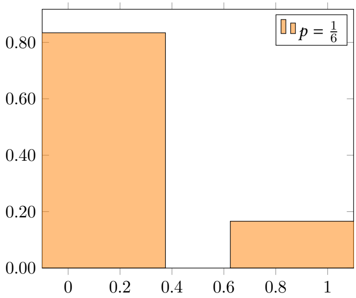

Binomial Distribution

Let $X_1, \dots, X_n$ be a finite sequence of random variables that each have a Bernoulli distribution with the same parameter p (finite Bernoulli process) and let $X := \sum_{i=1}^{n} X_i$. Then X has a binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$ ($X \sim B(n,p)$) and

$$\Pr[X = k] = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k \in \{0, 1, \dots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

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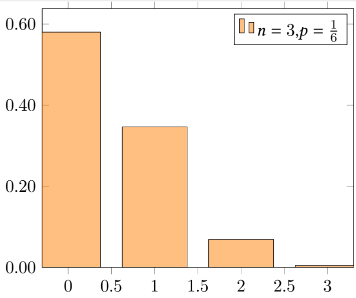

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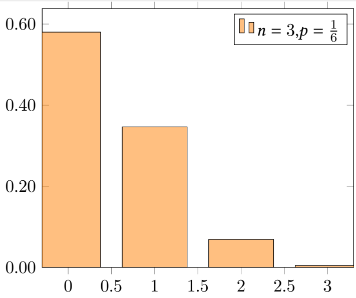

Poisson binomial distribution: arbitrary success parameters $p_1, p_2, \dots, p_n$

Geometric Distribution

Let $X_1, X_2, \ldots$ be a countably infinite sequence of independent random variables that each have a Bernoulli distribution with the same parameter p (infinite Bernoulli process) and let $X := \min\{i \geq 1 \mid X_i = 1\}$. Then X has a geometric distribution with parameter $p \in [0,1]$ ($X \sim G(p)$) and

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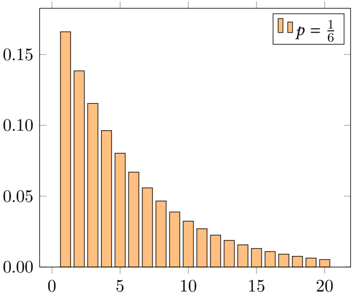

Expectation of Geometric Distribution

Theorem

Let X have a geometric distribution with parameter p. Then $\mathrm{Ex}[X] = \frac{1}{p}$.

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\end{aligned}$$

- **Thus:** $\mathrm{Ex}[X] = (1-p)\,\mathrm{Ex}[X] + 1$, also $\mathrm{Ex}[X] = \frac{1}{p}$

Memorylessness of Geometric Distribution

Theorem (Memorylessness of Geometric Distribution)

For every random variable X that has a geometric distribution and all $m, n \geq 0$ it holds that

$$\Pr[X > m + n \mid X > m] = \Pr[X > n] \, .$$

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Example

X: Number of die rolls until a six comes

$\Pr[X > 3 \mid X > 2] = \Pr[X > 2 + 1 \mid X > 2] = \Pr[X > 1] = 1 - p.$

Markov's Inequality

Theorem (Markov's Inequality)

Let X be a non-negative random variable with expected value $\mu := \mathrm{Ex}[X]$ and let $\alpha > 0$. Then

$$\Pr[X \geq \alpha \cdot \mu] \leq \frac{1}{\alpha} .$$

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With $\mu = \frac{7}{2}$ Markov's inequality gives: $\Pr[X \geq 4] = \Pr[X \geq \frac{8}{7} \cdot \mu] \leq \frac{7}{8}$.

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Actually we have: $\Pr[X \geq 4] = \frac{1}{2}$.

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Let X be a random variable with expected value $\mu := \mathrm{Ex}[X]$ and finite, non-zero standard deviation $\sigma = \sqrt{\mathrm{Var}[X]}$ and let $\alpha > 0$. Then

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Example

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With $\mu = \frac{7}{2}$ and $\sigma = \sqrt{\frac{35}{12}}$ Chebyshev's inequality gives:

$\Pr[X = 1 \vee X = 6] = \Pr[|X - \mu| \geq \frac{5}{2}]$

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Actually we have: $\Pr[X = 1 \vee X = 6] = \frac{1}{3}$.